ELSEVIER
Journal of Geometry and Physics 34 (2000) 121–136
JOURNAL OF GEOMETRY AND PHYSICS

Dynamics of the *n*-dimensional Suslov problem

Dmitry V. Zenkov*, Anthony M. Bloch
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Received 14 April 1999; received in revised form 18 August 1999

Abstract

In this paper we study the dynamics of a constrained generalized rigid body (the Suslov problem) on its full phase space. We use reconstruction theory to analyze the qualitative dynamics of the system and discuss differences with the free rigid body motion. Use is made of the so-called quasi-periodic Floquet theory.

MSC: 70H33; 58F27

Keywords: Reconstruction; Quasi-periodic flow; Nonholonomic systems

1. Introduction

In this paper we analyze the dynamics, and in particular the reconstruction of the so-called *Suslov problem* — a generalized rigid body with some of its body angular velocity components set equal to zero. The *unconstrained* generalized rigid body problem can be shown to be noncommutatively integrable and hence the system evolves on a torus of dimension lower than half that of the phase space. While the reduced (Euler–Poincaré–Suslov) dynamics of the Suslov problem is integrable (in a nonholonomic sense) there is no analogue of noncommutative integrability in the nonholonomic setting. The question then arises of what is the dynamics in the full space and in particular what is the reconstruction of the torus dynamics of the reduced equations.

We recall briefly the notions of reduction of mechanical systems. For a mechanical system with symmetry, the process of reduction neglects the directions along the group variables and thus provides a system with fewer degrees of freedom. In many important examples, the reduced system is integrable. Switching back to the original system

* Corresponding author.
E-mail addresses: zenkov@math.lsa.umich.edu (D. V. Zenkov), abloch@math.lsa.umich.edu (A. M. Bloch)

0393-0440/00/$ – see front matter
PII: S0393-0440(99)00058-3

is called reconstruction. If the symmetry group is abelian, then the reconstruction may be performed explicitly. The process of reconstruction in general, when the symmetry group is nonabelian, involves integration of a linear nonautonomous differential equation on a Lie group (see [12] for details). Even if we cannot explicitly perform the reconstruction, we still may obtain important characteristics of the motion, such as frequencies, for instance.

The relative equilibria and relative periodic orbits of flows with symmetry have been discussed by Ashwin and Melbourne [1]. They show that the closure of a reconstructed relative equilibrium (relative periodic orbit), if compact, is an invariant quasi-periodic torus. A natural generalization of a relative periodic orbit is a relative quasi-periodic orbit and below we generalize their approach to this setting. We consider a system with an integrable quasi-periodic reduced flow. We study the reconstruction equation and show that in certain situations its solutions are quasi-periodic or may be approximated by quasi-periodic curves in the phase space. We then apply the theory to the integrable nonholonomic Suslov problem.

We intend in a future work to use the results obtained in this paper to study more general problems of integrable nonholonomic systems.

2. Equations of motion of nonholonomic systems with symmetries

In this section, we briefly discuss the dynamics of nonholonomic systems with symmetries. We refer the reader to [3,15] for a more complete exposition.

2.1. The Lagrange–d'Alembert principle

We now describe the equations of motion for a nonholonomic system. We confine our attention to nonholonomic constraints that are homogeneous in the velocity. Accordingly, we consider a configuration space Q and a distribution $\mathcal{D}$ that describes these constraints. Recall that a distribution $\mathcal{D}$ is a collection of linear subspaces of the tangent spaces of Q; we denote these spaces by $\mathcal{D}_q \subset T_q Q$, one for each $q \in Q$. A curve $q(t) \in Q$ will be said to *satisfy the constraints* if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. This distribution will, in general, be nonintegrable, i.e. the constraints are, in general, nonholonomic.

Consider a Lagrangian $L : TQ \to \mathbb{R}$. In coordinates q^i, $i = 1, \dots, n$, on Q with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we write $L(q^i, \dot{q}^i)$. The equations of motion are given by the following Lagrange–d'Alembert principle.

Definition 2.1. The *Lagrange–d'Alembert equations of motion* for the system are those determined by

$$\delta \int_a^b L(q^i, \dot{q}^i)\,\mathrm{d}t = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t) \in \mathcal{D}_{q(t)}$ for each t where $a \le t \le b$.

This principle is supplemented by the condition that the curve itself satisfies the constraints. Note that we take the variation *before* imposing the constraints, i.e. we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations (see [3] for a discussion and references).

The usual arguments in the calculus of variations show that the Lagrange–d'Alembert principle is equivalent to the equations

$$-\delta L = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0 \tag{2.1}$$

for all variations δq such that $\delta q \in \mathcal{D}_q$ at each point of the underlying curve $q(t)$. One can of course equivalently write these equations in terms of Lagrange multipliers.

Let $\{\omega^a, a = 1, \dots, p\}$ be a set of p independent one forms whose vanishing describes the constraints. Choose a local coordinate chart $q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, which we write as $q^i = (r^\alpha, s^a)$, where $1 \le \alpha \le n - p$ and $1 \le a \le p$ such that

$$\omega^a(q) = \mathrm{d}s^a + A^a_\alpha(r, s)\, \mathrm{d}r^\alpha$$

for all $a = 1, \dots, p$. In these coordinates, the constraints are described by vectors $v^i = (v^\alpha, v^a)$ satisfying $v^a + A^a_\alpha v^\alpha = 0$ (a sum on repeated indices over their range is understood).

The equations of motion for the system are given by (2.1) where we choose variations $\delta q(t)$ that satisfy the constraints, i.e., $\omega^a(q)\delta q = 0$, or equivalently, $\delta s^a + A^a_\alpha \delta r^\alpha = 0$, where $\delta q^i = (\delta r^\alpha, \delta s^a)$. Substituting variations of this type, with δr^α arbitrary, into (2.1) gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} \right) = A^a_\alpha \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a} \right) \tag{2.2}$$

for all $\alpha = 1, \dots, n - p$. Eqs. (2.2), combined with the constraint equations

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha \tag{2.3}$$

for all $a = 1, \dots, p$, give the complete equations of motion of the system.

A useful way of reformulating Eqs. (2.2) is to define a *constrained Lagrangian* by substituting the constraints (2.3) into the Lagrangian:

$$L_c(r^\alpha, s^a, \dot{r}^\alpha) := L(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r, s)\dot{r}^\alpha).$$

The equations of motion can be written in terms of the constrained Lagrangian in the following way, as a direct coordinate calculation shows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta} \dot{r}^\beta,$$

where $B^b_{\alpha\beta}$ is defined by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{A^b_\alpha}{\partial s^a} \right).$$

Geometrically, the A^a_α are the coordinate expressions for the Ehresmann connection on the tangent bundle defined by the constraints, while the $B^b_{\alpha\beta}$ are the corresponding curvature terms (see [3]).

2.2. Symmetries

As we shall see shortly, symmetries play an important role in our analysis. We begin here with just a few preliminary notions. Suppose we are given a nonholonomic system with Lagrangian $L : TQ \to \mathbb{R}$, and a (nonintegrable) constraint distribution $\mathcal{D}$. We can then look for a group G that acts on the configuration space Q. It induces an action on the tangent space TQ and so it makes sense to ask that the Lagrangian L be *invariant*. Also, one can ask that the distribution be *invariant* in the sense that the action by a group element $g \in G$ maps the distribution $\mathcal{D}_q$ at the point $q \in Q$ to the distribution $\mathcal{D}_{gq}$ at the point gq. If these properties hold, we say that G is a *symmetry group*.

2.3. The geometry of nonholonomic systems with symmetry

Consider a nonholonomic system with the Lagrangian $L : TQ \to \mathbb{R}$, the (nonintegrable) constraint distribution $\mathcal{D}$, and the symmetry group G in the sense explained previously.

Orbits and shape space. The group orbit through a point q, an (immersed) submanifold, is denoted

$$\mathrm{Orb}(q) := \{gq | g \in G\}.$$

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G. For an element $\xi \in \mathfrak{g}$, we write ξ_Q, a vector field on Q for the corresponding infinitesimal generator, so these are also the tangent spaces to the group orbits. Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in both $\mathcal{D}_q$ and $T_q(\mathrm{Orb}(q))$:

$$\mathfrak{g}^q := \left\{\xi \in \mathfrak{g} | \xi_Q(q) \in \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))\right\}.$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$, is denoted by $\mathfrak{g}^{\mathcal{D}}$.

Reduced dynamics. Assuming that the Lagrangian and the constraint distribution are G-invariant, we can form the *reduced velocity phase space* TQ/G and the *reduced constraint space* $\mathcal{D}/G$. The Lagrangian L induces well defined functions, the *reduced Lagrangian*

$$l : TQ/G \to \mathbb{R}$$

satisfying $L = l \circ \pi_{TQ}$ where $\pi_{TQ} : TQ \to TQ/G$ is the projection, and the *constrained reduced Lagrangian*

$$l_c : \mathcal{D}/G \to \mathbb{R},$$

which satisfies $L|_{\mathcal{D}} = l_c \circ \pi_{\mathcal{D}}$ where $\pi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}/G$ is the projection. By general considerations, the Lagrange–d'Alembert equations induce well defined *reduced equations* on $\mathcal{D}/G$, i.e. the vector field on the manifold $\mathcal{D}$ determined by the Lagrange–d'Alembert equations (including the constraints) is G-invariant, and so defines a reduced vector field on the quotient manifold $\mathcal{D}/G$. Following [5], we call these equations the *Lagrange–d'Alembert–Poincaré equations.*

Let a local trivialization be chosen on the principle bundle $\pi : Q \to Q/G$, with a local representation having components denoted (r, g). Let r, an element of shape space Q/G, have coordinates denoted r^α, and let g be group variables for the fiber, G. In such a representation, the action of G is the left action of G on the second factor. The coordinates (r, g) induce the coordinates $(r, \dot{r}, \xi)$ on TQ/G, where $\xi = g^{-1}\dot{g}$. The Lagrangian L is invariant under the left action of G and so it depends on g and $\dot{g}$ only through the combination $\xi = g^{-1}\dot{g}$. Thus the reduced Lagrangian l is given by

$$l(r, \dot{r}, \xi) = L(r, g, \dot{r}, \dot{g}).$$

Therefore, the full system of equations of motion consists of the following two groups:
1. The Lagrange–d'Alembert–Poincaré equation on $\mathcal{D}/G$ (see Theorem 2.2).
2. The *reconstruction equation*

$$\dot{g} = g\xi.$$

The nonholonomic momentum in body representation. Choose a q-dependent basis $e_a(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$ in the following way. First, one chooses, for each r, such a basis at the identity element $g = \mathrm{Id}$, say

$$e_1(r), e_2(r), \dots, e_m(r), e_{m+1}(r), \dots, e_k(r).$$

For example, this could be a basis whose generators are orthonormal in the kinetic energy metric. Now define the *body fixed basis* by

$$e_a(r, g) = \mathrm{Ad}_g\, e_a(r),$$

then the first m elements will indeed span the subspace $\mathfrak{g}^q$ since the distribution is invariant.

To avoid confusion, we will make the following index and summation conventions:
1. The first batch of indices range from 1 to m corresponding to the symmetry directions along constraint space. These indices will be denoted $a, b, c, d, \dots$ and a summation from 1 to m will be understood.
2. The second batch of indices range from $m + 1$ to k corresponding to the symmetry directions not aligned with the constraints. Indices for this range or for the whole range 1 to k will be denoted by $a', b', c', \dots$ and the summations will be given explicitly.
3. The indices $\alpha, \beta, \dots$ on the shape variables r range from 1 to σ. Thus, σ is the dimension of the shape space Q/G and so $\sigma = n - k$. The summation convention for these indices will be understood.

Assume that the Lagrangian has the form kinetic energy minus potential energy, and that the constraints and the orbit directions span the entire tangent space to the configuration space:

$$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_q Q.$$

Then it is possible to introduce a new Lie algebra variable Ω called the *body angular velocity* such that

1. $\Omega = \mathcal{A}\dot{r} + \xi$, where the operator $\mathcal{A}$ is called the *nonholonomic connection*.
2. The constraints are given by $\Omega \in \operatorname{span}\{e_1(r), \dots, e_m(r)\}$ or $\Omega^{m+1} = \cdots = \Omega^k = 0$.
3. The reduced Lagrangian in the variables $(r, \dot{r}, \Omega)$ becomes

$$\frac{1}{2} g_{\alpha\beta}\dot{r}^\alpha \dot{r}^\beta + \frac{1}{2} I_{ac}\Omega^a \Omega^c + \sum_{a'=m+1}^{k} \left(l_{a'\alpha} - l_{a'c'}\mathcal{A}_\alpha^{c'} \right) \Omega^{a'} \dot{r}^\alpha + \frac{1}{2} \sum_{a',c'=m+1}^{k} l_{a'c'}\Omega^{a'}\Omega^{c'} - V. \tag{2.4}$$

In the above, $g_{\alpha\beta}$ are coefficients of the kinetic energy metric induced on the manifold Q/G, I_{ac} are components of the *locked inertia tensor* relative to $\mathfrak{g}^{\mathcal{D}}$, $\mathbb{I}(q) : \mathfrak{g}^{\mathcal{D}} \to \left(\mathfrak{g}^{\mathcal{D}}\right)^*$ defined by

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\langle \xi_Q, \eta_Q \rangle\rangle, \qquad \xi, \eta \in \mathfrak{g}^q,$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the kinetic energy metric, and

$$l_{a'\alpha} = \frac{\partial^2 l}{\partial \xi^{a'} \partial \dot{r}^\alpha}, \qquad l_{a'c'} = \frac{\partial^2 l}{\partial \xi^{a'} \partial \xi^{c'}}.$$

We remark that this choice of Ω eliminates the terms proportional to $\Omega^a \dot{r}^\alpha$ and $\Omega^a \Omega^{a'}$ in (2.4). The constrained reduced Lagrangian becomes especially simple in variables $(r, \dot{r}, \Omega)$:

$$l_c = \tfrac{1}{2} g_{\alpha\beta}\dot{r}^\alpha \dot{r}^\beta + \tfrac{1}{2} I_{ac}\Omega^a \Omega^c - V.$$

The *nonholonomic momentum in body representation* is defined by

$$p_a = \frac{\partial l}{\partial \Omega^a} = \frac{\partial l_c}{\partial \Omega^a}.$$

Notice that the nonholonomic momentum may be viewed as a collection of components of the ordinary momentum map along the constraint directions.

The Lagrange–d'Alembert–Poincaré equations. As in [3], the reduced equations of motion are given by the next theorem.

Theorem 2.2. *The following nonholonomic Lagrange–d'Alembert–Poincaré equations hold for each* $1 \le \alpha \le \sigma$ *and* $1 \le b \le m$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} = -\frac{\partial I^{cd}}{\partial r^\alpha} p_c p_d - \mathcal{D}_{b\alpha}^c I^{bd} p_c p_d - \mathcal{B}_{\alpha\beta}^c p_c \dot{r}^\beta - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma}\dot{r}^\beta \dot{r}^\gamma, \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_b = C_{ab}^c I^{ad} p_c p_d + \mathcal{D}_{b\alpha}^c p_c \dot{r}^\alpha + \mathcal{D}_{\alpha\beta b}\dot{r}^\alpha \dot{r}^\beta. \tag{2.6}$$

Here $l_c(r^\alpha, \dot{r}^\alpha, p_a)$ *is the constrained Lagrangian;* r^α, $1 \le \alpha \le \sigma$, *are coordinates in the shape space;* p_a, $1 \le a \le m$, *are components of the momentum map in the body* *representation; I^{ad} are the components of the inverse locked inertia tensor; $\mathcal{B}^a_{\alpha\beta}$ are the local coordinates of the curvature $\mathcal{B}$ of the nonholonomic connection $\mathcal{A}$; the coefficients $\mathcal{D}^c_{b\alpha}$, $\mathcal{D}_{\alpha\beta b}$, $\mathcal{K}_{\alpha\beta\gamma}$ are given by the formulae*

$$\mathcal{D}^c_{b\alpha} = \sum_{a'=1}^{k} - C^c_{a'b}\mathcal{A}^{a'}_\alpha + \gamma^c_{b\alpha} + \sum_{a'=m+1}^{k} \lambda_{a'\alpha} C^{a'}_{ab} I^{ac},$$

$$\mathcal{D}_{\alpha\beta b} = \sum_{a'=m+1}^{k} \lambda_{a'\alpha}\left(\gamma^{a'}_{b\beta} - \sum_{b'=1}^{k} C^{a'}_{b'b}\mathcal{A}^{b'}_\beta\right), \quad \mathcal{K}_{\alpha\beta\gamma} = \sum_{a'=1}^{k}\lambda_{a'\gamma}\mathcal{B}^{a'}_{\alpha\beta},$$

where

$$\lambda_{a'\alpha} = l_{a'\alpha} - \sum_{b'=1}^{k} l_{a'b'}\mathcal{A}^{b'}_\alpha := \frac{\partial l}{\partial \xi^{a'}\partial \dot{r}^\alpha} - \sum_{b'=1}^{k}\frac{\partial l}{\partial \xi^{a'}\partial \xi^{b'}}\mathcal{A}^{b'}_\alpha$$

for $a' = m+1, \ldots, k$. Here $C^{b'}_{a'c'}$ are the structure constants of the Lie algebra defined by $[e_{a'}, e_{c'}] = C^{b'}_{a'c'} e_{b'}$, $a', b', c' = 1, \ldots, k$; the coefficients $\gamma^{c'}_{b\alpha}$ are defined by

$$\frac{\partial e_b}{\partial r^\alpha} = \sum_{c'=1}^{k}\gamma^{c'}_{b\alpha} e_{c'}.$$

We shall discuss in detail the nonholonomic system of interest here, the Suslov problem, after a general discussion of relative equilibria and reconstruction theory.

3. Relative equilibria and relative periodic orbits

In this section we discuss how the relative equilibria and relative periodic orbits of flows with symmetries are related to maximal tori of the symmetry group. We follow the exposition of [1].

Consider a vector field $X(x)$ on a manifold M. Let G be a Lie group acting on the manifold M. For simplicity we assume that the orbit space M/G is a smooth manifold. Suppose that the vector field $X(x)$ is G-invariant, and that the flow $F_t : M \to M$ of this vector field is complete. The flow F_t induces a *reduced flow* $\phi_t : M/G \to M/G$, $\phi_t = \pi \circ F_t$, where $\pi : M \to M/G$ is the projection. We have the following definition [13]:

Definition 3.1. An orbit $\gamma(t)$ is called a *relative equilibrium* (a *relative periodic orbit*) if the orbit $\pi \circ \gamma(t)$ of the reduced flow is an equilibrium (a periodic orbit).

Remark. *All orbits $\gamma(t)$ generated by the same relative equilibrium (relative periodic orbit) belong to the same group orbit*

$$\mathrm{Orb}(\pi \circ \gamma(t)) = \{\mathrm{Orb}(x) | x \in \pi \circ \gamma(t)\},$$

which sometimes is also referred to as a relative equilibrium (relative periodic orbit respectively) (*see* [1]).

Suppose that the group G is connected. The following two theorems from [1] explain what the reconstructed relative equilibria and relative periodic orbits are. Similar results may be found in [7,11,13].

Theorem 3.2 (Ashwin and Melbourne [1]). *Suppose that the relative equilibrium x_e has isotropy subgroup Σ. Then the group orbit through x_e is foliated by tori T^p, $p \leq \operatorname{rank}(N(\Sigma)/\Sigma)$, or by copies of $\mathbb{R}$.*

In this theorem, $N(\Sigma)$ denotes the normalizer of Σ. Next, consider a system with a relative periodic orbit $\gamma(t)$. Let $x_0 = \gamma(0)$ and let Σ be the isotropy of x_0.

Theorem 3.3 (Ashwin and Melbourne [1]). *A group orbit through the relative periodic orbit is foliated by tori T^p, $p \leq \operatorname{rank}(N(\Sigma)/\Sigma) + 1$, with irrational tori flow or by copies of $\mathbb{R}$ with unbounded linear flow.*

In the above theorems, the upper bound for the dimension of the invariant tori is attained generically. If the isotropy Σ is a trivial subgroup of G, then $p \leq \operatorname{rank} G$ ($p \leq \operatorname{rank} G + 1$, respectively).

4. Relative quasi-periodic orbits

In this section we discuss the reconstruction process applied to relative quasi-periodic orbits.

Systems with quasi-periodic reduced dynamics. Suppose that the reduced flow of the mechanical system with symmetry group G is *quasi-periodic*, i.e. the reduced phase space is foliated by m-dimensional tori and the flow on these tori is

$$\dot{\phi}_1 = \omega_1, \ldots, \dot{\phi}_m = \omega_m,$$

where the frequencies ω_j are incommensurate (i.e. are not rationally related). In this case the Lie algebra element $\xi(t)$ in the reconstruction equation

$$\dot{g} = g\xi(t) \tag{4.1}$$

is a quasi-periodic function, i.e.

$$\xi(t) = \Xi(\omega_1 t, \ldots, \omega_m t)$$

for an appropriate function $\Xi : T^m \to \mathfrak{g}$. If the symmetry group is abelian (and compact), then this equation is explicitly solvable and the generic motion is quasi-periodic on $(m + \dim G)$-dimensional tori.

Suppose that the reconstruction equation is *reducible*, i.e. there exists a substitution $g = ha(t)$, where $a(t)$ is a quasi-periodic group element such that the Lie algebra element

$$\eta = a(t)\xi(t)a^{-1}(t) - \dot{a}(t)a^{-1}(t)$$

is time-independent.

Theorem 4.1. *If the reconstruction equation is reducible, then the trajectories* $(r(t), g(t))$ *of the original system are quasi-periodic.*

Proof. The reconstruction equation after the substitution $g = ha(t)$ becomes

$$\dot{h} = h\eta.$$

Since η is a fixed Lie algebra element, the solutions $h(t) = h_0 \exp(t\eta)$ are quasi-periodic. By Theorem 3.2 for a generic η, $\overline{h(t)}$ is a $\operatorname{rank}(N(\Sigma)/\Sigma)$-dimensional torus. The corresponding trajectory of the system $(r(t), g(t))$ is thus quasi-periodic. Generically, it has $(\operatorname{rank}(N(\Sigma)/\Sigma)+m)$ frequencies, where m is the number of frequencies of the quasi-periodic Lie algebra element $\xi(t)$. If the group G acts on the phase space without fixed points, then the trajectory $(r(t), g(t))$ has $(\operatorname{rank} G + m)$ frequencies. □

The main question therefore is whether the equation $\dot{g} = g\xi(t)$ is reducible. Suppose that the group G is a matrix group, then Eq. (4.1) is reducible if and only if the linear system of differential equations

$$\dot{x} = \xi^{\mathrm{T}}(t)x$$

is reducible.

Quasi-periodic Floquet theory. Consider a system of linear differential equations

$$\dot{x} = A(t)x. \tag{4.2}$$

If the matrix $A(t)$ is periodic, then the Floquet theory states that Eq. (4.2) is reducible. This means that there exists a periodic linear substitution $x = C(t)y$ such that Eq. (4.2) becomes

$$\dot{y} = By,$$

where $B = C^{-1}(t)A(t)C(t) - C^{-1}(t)\dot{C}(t)$ is a *constant* matrix.

However, if the matrix $A(t)$ is quasi-periodic, system (4.2) may be irreducible (see [9] and references therein). We remark here that the conditions for reducibility are either "nonconstructive" (like those of [9]), or too restrictive. For instance, the reducibility conditions in [4] require that the eigenvalues of the constant part of the matrix A satisfy certain conditions. These conditions do not hold for many interesting mechanical examples, such as the Suslov problem.

Effective reducibility. The following theorem (see [8]) states that the quasi-periodic part of the matrix $A(t)$ can be made exponentially small. As before, $\omega_1, \ldots, \omega_m$ are the frequencies of the quasi-periodic matrix.

Theorem 4.2. *Consider the equation* $\dot{x} = (A + \varepsilon Q(t, \varepsilon))x,\ \ |\varepsilon| < \varepsilon_0,\ \ x \in \mathbb{R}^d$, *where we have the following hypotheses:*

1. *A is a constant $d \times d$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_d$.*
2. *$Q(t, \varepsilon)$ is an analytic quasi-periodic matrix on a strip of width ρ with $\|Q(\cdot, \varepsilon)\|_\rho \le q$ for all $|\varepsilon| \le \varepsilon_0$, for some $\omega \in \mathbb{R}^m$, where $q, \rho > 0$.*
3. *The vector ω satisfies the diophantine conditions*

$$|\lambda_j - \lambda_l + \mathrm{i}(k,\omega)| \geq \frac{c}{|k|^\gamma}$$

for any $k \in \mathbb{Z}^m \backslash 0$ and for any $j, l \in \{1, \dots, d\}$ for some constants $c > 0$ and $\gamma > m-1$. As usual, $|k| = |k_1| + \cdots + |k_m|$.

Then there exist positive constants ε^*, a^*, r^*, and μ such that for all ε, $|\varepsilon| \leq \varepsilon^*$, the initial equation can be transformed into

$$\dot{y} = (A^*(\varepsilon) + \varepsilon R^*(t,\varepsilon))y,$$

where

1. *A^* is a constant matrix with $|A^*(\varepsilon) - A|_\infty \leq a^*|\varepsilon|$ and*
2. *$R^*(\cdot,\varepsilon)$ is an analytic quasi-periodic function on a strip of width ρ with $\|R^*(\cdot,\varepsilon)\|_\rho \leq r^* \exp(-(\mu/|\varepsilon|)^{1/\gamma}\delta)$ for any $\delta \in (0,\rho]$.*

Furthermore, the quasi-periodic change of variables that performs this transformation is also analytic on a strip of width ρ.

So, if $\xi(t) = A + \varepsilon Q(t,\varepsilon)$, then the reconstruction equation reduces to

$$\dot{h} = h(A^*(\varepsilon) + \varepsilon R^*)$$

with an exponentially small quasi-periodic term. Therefore, the flow determined by the reconstruction equation may be approximated by a quasi-periodic flow on a time interval of length $\sim \exp(1/\varepsilon)$. We remark that this is a considerably longer time span than the time span of order $1/\varepsilon$ that would be expected from standard perturbation theory arguments.

In the example below (the Suslov problem), the constant matrix A has multiple zero eigenvalues. Theorem 4.2 is valid for multiple eigenvalues too, but the exponent of ε in the expression for R^* is slightly worse (see [8]).

5. The Suslov problem

In this section we study the constrained dynamics of the Euler top. The constraints require that some of the components of the body angular velocity are equal to zero.

5.1. The classical Suslov problem

The Suslov problem is an Euler top with a nonholonomic constraint $\langle a, \Omega\rangle = 0$, where Ω is the body angular velocity and a is a vector fixed in body. Here $\langle\cdot,\cdot\rangle$ stands for the standard metric in $\mathbb{R}^3$. The configuration space of this system is the group $SO(3)$. The Lagrangian (the kinetic energy) and the constraint are invariant under the left action of $SO(3)$ on the configuration space.

The Suslov problem belongs to a class of nonholonomic systems with no shape space. The Lagrange–d'Alembert–Poincaré equations (see Theorem 2.2) for such systems reduce to the momentum equation

$$\dot{p}_b = C^c_{ab} I^{ad} p_c p_d, \tag{5.1}$$

where p_a are the components of the nonholonomic momentum relative to the body frame, I^{ad} are the components of the inverse inertia tensor, and C^c_{ab} are the structure constants of $\mathfrak{g}$. Eq. (5.1) are not in general Euler–Poincaré equations because the subspace $\mathfrak{g}^{\mathcal{D}} = \{\Omega \mid \langle a, \Omega\rangle = 0\}$ is not necessarily a subalgebra. Instead, we may view (5.1) as Euler–Poincaré equations restricted to a subspace $\left(\mathfrak{g}^{\mathcal{D}}\right)^*$ of the Lie coalgebra $\mathfrak{g}^*$. Following Fedorov and Kozlov we call these the Euler–Poincaré–Suslov equations.

Choose $e_3 = a/|a|$ as the third vector of the body frame. Then the constraint becomes $\Omega^3 = 0$. Pick two independent vectors e_1 and e_2 that are orthogonal to e_3 in the *kinetic energy metric*. These vectors e_1, e_2, and e_3 *are not orthogonal* relative to the standard metric in $so(3) = \mathbb{R}^3$ unless e_3 spans an eigenspace of the inertia tensor. Consequently, the structure constants C^1_{12} and C^2_{12} are not necessarily equal to zero. In the frame e_1, e_2, e_3 the components I_{13} and I_{23} of the inertia tensor are equal to zero. Therefore

$$p_3 = \frac{\partial l_c}{\partial \Omega^3} = I_{33}\Omega^3 = 0$$

and Eqs. (5.1) become

$$\dot{p}_1 = \left(C^1_{21}p_1 + C^2_{21}p_2\right)\left(I^{12}p_1 + I^{22}p_2\right), \tag{5.2}$$

$$\dot{p}_2 = \left(C^1_{12}p_1 + C^2_{12}p_2\right)\left(I^{11}p_1 + I^{12}p_2\right). \tag{5.3}$$

These equation are equivalent to the equations in [6].

It is known that Eqs. (5.2) and (5.3) are not integrable unless e_3 is an eigenvector of the locked inertia tensor (see [10] for details). In this last case $\mathbb{I}e_3 = I_3e_3$ and we may choose the two remaining eigenvectors to be e_1 and e_2 as defined above, so that the basis e_1, e_2, e_3 is orthogonal with respect to both the standard and the kinetic energy metrics. In this basis, $C^1_{12} = C^2_{12} = 0$. Hence in the integrable case Eqs. (5.2) and (5.3) become

$$\dot{p}_1 = 0, \qquad \dot{p}_2 = 0.$$

Thus all the solutions of the reduced system are relative equilibria. By Theorem 3.2, almost all reconstructed motions of the classical Suslov problem are periodic because the rank of the group $SO(3)$ is equal to one.

5.2. The n-dimensional Suslov problem

Fedorov and Kozlov [6] considers an n-dimensional Suslov problem, i.e. they consider the motion of an n-dimensional rigid body with a diagonal inertia tensor $I_{ij} = \delta_{ij}I_j$, $I_1 > I_2 > \cdots > I_n$, subject to the constraints $\Omega_{ij} = 0,\ \ i, j > 2$, where $\Omega \in so(n)$ is the body angular velocity. This system is $SO(n)$-invariant, where the group $SO(n)$ acts on the configuration space by left shifts.

Integrability of the reduced flow. Fedorov and Kozlov [6] prove that the reduced system on $so(n)$ has $n-1$ quadratic integrals

$$F_0(\Omega) = h/2, \quad F_1(\Omega) = c_1, \ldots, F_{n-2}(\Omega) = c_{n-2},$$

where $F_0(\Omega)$ is the positive-definite energy integral. The common level surface of these integrals is diffeomorphic to the disjoint union of $(n-2)$-dimensional tori if $h, c_1, \dots, c_{n-2}$ are all positive and satisfy the condition

$$\frac{c_1}{I_2 - I_3} + \cdots + \frac{c_{n-2}}{I_2 - I_n} < h. \tag{5.4}$$

The flow on these tori in the appropriate angular coordinates is governed by the equations

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}\tau} = \omega_1, \quad \dots, \quad \frac{\mathrm{d}\phi_{n-2}}{\mathrm{d}\tau} = \omega_{n-2}, \tag{5.5}$$

where τ is a new independent variable introduced by $\mathrm{d}\tau = \Omega_{12}\,\mathrm{d}t$. The solutions of Eqs. (5.5) are quasi-periodic motions on tori. The reduced Suslov problem is therefore integrable. The frequencies $\omega_1, \dots, \omega_{n-2}$ are the same for all the tori and depend only on the values of $I_1, \dots, I_n$. Explicitly,

$$\omega_s = \sqrt{f(I_{s+2})}, \qquad f(z) = \frac{(I_1 - z)(I_2 - z)}{(I_1 + z)(I_2 + z)}.$$

The function $f(z)$ is decreasing on the interval $[0, I_2]$ and takes values between 0 and 1. Therefore the set of $(I_1, \dots, I_n)$ for which the diophantine conditions (5.7) fail is of zero measure.

The nonzero components of the body angular velocity are

$$\Omega_{12} = \frac{1}{I_1 + I_2}\sqrt{h - \sum_{s=1}^{n-2}\left(\frac{c_s}{I_2 - I_{s+2}}\sin^2\phi_s + \frac{c_s}{I_1 - I_{s+2}}\cos^2\phi_s\right)},$$

$$\Omega_{1,s+2} = \sqrt{\frac{c_s}{(I_1 + I_{s+2})(I_2 - I_{s+2})}}\sin\phi_s,$$

$$\Omega_{2,s+2} = \sqrt{\frac{c_s}{(I_2 + I_{s+2})(I_1 - I_{s+2})}}\cos\phi_s.$$

If the trajectories are closed (periodic) on one torus, then they are closed on the rest of the tori as well. If all the trajectories are closed, we can view the reduced dynamics as a collection of relative periodic orbits organized in invariant $(n-2)$-dimensional tori. By Theorem 3.3 each relative periodic orbit gives rise to the $(r+1)$-dimensional quasi-periodic invariant torus in $\mathcal{D}$. Here $r \leq \operatorname{rank}(SO(n))$. Since the relative periodic orbits are organized in $(n-2)$-dimensional tori, the invariant manifolds are $(r+n-2)$-dimensional tori foliated by quasi-periodic $(r+1)$-dimensional tori.

Quasi-periodic reconstruction. Assume that the conditions given in the previous paragraph hold. Then the reduced dynamics is quasi-periodic on $(n-2)$-dimensional tori in the Lie algebra $so(n)$. The equations of motion of the Suslov problem consist of the reduced system on the algebra $so(n)$ (this system is integrable) coupled with the reconstruction equation $\dot{g} = g\Omega$. Recall that the frequencies $\omega_1, \dots, \omega_{n-2}$ are completely determined by the inertia tensor and are the same for all invariant tori of the reduced system.

To see what the motions in the full phase space are, we need to perform the reconstruction, i.e. to solve the equation

$$\dot{g} = g\Omega(t).$$

The solutions of the reduced system become simpler if we introduce a new independent variable τ by $\mathrm{d}\tau = \Omega_{12}\,\mathrm{d}t$. The reconstruction equation becomes

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = g\xi(\tau), \tag{5.6}$$

where ξ is a skew-symmetric matrix with components

$$\xi_{ij} = \frac{\Omega_{ij}}{\Omega_{12}}.$$

In particular, $\xi_{12} = -\xi_{21} = 1$. Since Ω_{ij} are quasi-periodic functions of τ, the matrix $\xi(\tau)$ is quasi-periodic (with frequencies $\omega_1, \dots, \omega_{n-2}$).

Reducibility of the reconstruction equation and symmetries. Here we discuss reducibility of the reconstruction Eq. (5.6) and symmetries of the generalized Suslov problem. Since the reduced flow, given by Eq. (5.5), is quasi-periodic, the reduced Suslov problem is $T^{(n-2)}$-invariant. The action is given by

$$T^{(n-2)} \ni (\alpha_1, \dots, \alpha_{n-2}) \mapsto (\phi_1 + \alpha_1, \dots, \phi_{n-2} + \alpha_{n-2}).$$

Theorem 5.1. *The reconstruction equation (5.6) is reducible iff the Suslov problem is* $SO(n) \times T^{(n-2)}$-invariant.

Proof. Recall that the equations of motion are

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \omega, \qquad \frac{\mathrm{d}g}{\mathrm{d}\tau} = g\xi(\tau),$$

where $F = (F_0, \dots, F_{n-2})$ are the integrals of the reduced Suslov problem. If this system is $SO(n) \times T^{(n-2)}$-invariant, then the reduced dynamics consists of relative equilibria $F =$ const. The reconstructed motions are therefore quasi-periodic. Since the group $SO(n)$ acts on itself without fixed points, generic reconstructed motions are represented by the flow on the maximal tori of $SO(n) \times T^{(n-2)}$. The dimension of these tori equals rank $(SO(n)) + (n-2)$. In particular, there exist coordinates on $SO(n) \times T^{(n-2)}$ in which the $SO(n) \times T^{(n-2)}$-reconstruction equations are

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \omega, \qquad \frac{\mathrm{d}h}{\mathrm{d}\tau} = h\xi,$$

where ξ is a *fixed* element of $so(n)$. Thus, the $SO(n)$-reconstruction equation is reducible.

Suppose now that the equation

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = g\xi(\tau)$$

is reducible. Then we can choose new coordinates h on the group $SO(n)$ such that the reconstruction equation becomes

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = h\xi, \qquad \xi = \text{const}.$$

The equations of motion thus become

$$\frac{\mathrm{d}F}{\mathrm{d}\tau}=0,\qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau}=\omega,\qquad \frac{\mathrm{d}h}{\mathrm{d}\tau}=h\xi.$$

These equations are $SO(n)\times T^{(n-2)}$-invariant. □

Remark. *Of course the inertia tensor contains the complete information about the dynamics of the Suslov problem. In particular, it determines in principle the $SO(n)\times T^{(n-2)}$-invariance/reducibility of the reconstruction equation. However, there are no adequate methods that allow one to tell when this reducibility occurs. To work around this difficulty we use below the effective reducibility approach. It is interesting to notice that the presence of the symplectic structure in the theory of the unconstrained rigid body allows one to construct a bigger symmetry group and to establish reducibility of the reconstruction equation. The reduced dynamics consists of the relative equilibria only. This is why the unconstrained n-dimensional rigid body is an integrable system. See* [14], *and for example,* [2] *for details and theory of noncommutative integrability. The absence of a symplectic structure and a suitable notion of commuting integrals in nonholonomic mechanics prevent us from showing that the Suslov problem is $SO(n)\times T^{(n-2)}$ -invariant.*

Effective reducibility and reconstruction. We consider two cases when the effective reducibility approach may be used.

Case 1. Here we consider the motions with one dominant component of the angular velocity. Put $c_s=\varepsilon b_s$, $s=1,\dots,n-2$. In this case $|\Omega_{l,s+2}|\ll|\Omega_{12}|$, $l=1,2$, $s=1,\dots,n-2$. Then the reconstruction equation takes the form

$$\frac{\mathrm{d}g}{\mathrm{d}\tau}=g(A+\varepsilon Q(\tau,\varepsilon)),$$

where A is a constant skew-symmetric matrix with entries $A_{12}=-A_{21}=1$ and $A_{ij}=0$ for all other pairs of indices i and j. Assume that the following diophantine conditions hold for some constants $c>0$ and $\gamma>n-3$:

$$|l+\mathrm{i}(k,\omega)|\geq\frac{c}{|k|^{\gamma}},\qquad l=0,\ 1,\ 2. \tag{5.7}$$

As we mentioned before, this is true for a generic inertia tensor. Even if the reconstruction equation is not reducible, we can apply Theorem 4.2. Condition 2 of this theorem follows from (5.4). We conclude thus that the trajectories of the Suslov problem may be approximated by quasi-periodic curves on the time interval of length $\sim\exp(1/\varepsilon)$.

Case 2. Now consider the case when $c_s=\varepsilon b_s$, $s=2,\dots,n-2$, i.e. we have motions with three dominant components, Ω_{12}, Ω_{13}, and Ω_{23}, of the angular velocity. Then the reconstruction equation is of the form

$$\frac{\mathrm{d}g}{\mathrm{d}\tau}=g(B(t)+\varepsilon Q(\tau,\varepsilon)),$$

where $B(t)$ is a $2\pi/\omega_1$-periodic function and $Q(\tau,\varepsilon)$ is a quasi-periodic function. We first find a periodic substitution $k = ga(t)$ which transforms the equation $\mathrm{d}g/\mathrm{d}\tau = gB(t)$ into $\mathrm{d}k/\mathrm{d}\tau = kB$. The reconstruction equation thus becomes

$$\frac{\mathrm{d}k}{\mathrm{d}\tau} = k(B + \varepsilon\tilde{Q}(\tau,\varepsilon)), \qquad B = \text{const}.$$

Assume that the appropriate diophantine conditions hold. Then, as above, we find a quasi-periodic substitution $h = kb(t)$ which makes quasi-periodic terms in the reconstruction equation exponentially small in ε. Thus, in the case of three dominant components of the angular velocity we observe the same behavior as in the case of one dominant component. Summarizing, we have:

Theorem 5.2. *Consider the n-dimensional Suslov problem with quasi-periodic reduced flow. In the case of one or three dominant components of the angular velocity, the dynamics of the n-dimensional Suslov problem may be approximated by quasi-periodic dynamics on the time interval of length* $\sim \exp(1/\varepsilon)$.

Acknowledgements

We would like to thank Professors A. Derdzinski, M.G. Forest, and J. Marsden for helpful discussions, and the reviewer for helpful remarks. Research partially supported by NSF grant DMS–9803181, AFOSR grant F49620-96-1-0100, an NSF group infrastructure grant at the University of Michigan (DVZ and AMB) and a University of Michigan Rackham Fellowship (DVZ).

References

[1] P. Ashwin, I. Melbourne, Noncompact drift for relative equilibria and relative periodic orbits, Nonlinearity 10 (1997) 595–616.
[2] A.M. Bloch, P.E. Crouch, J.E. Marsden, T.S. Ratiu, Discrete rigid body dynamics and optimal control, Proc. CDC 37 (1998) 2249–2254.
[3] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R. Murray, Nonholonomic mechanical systems with symmetry, Arch. Rat. Mech. Ann. 136 (1996) 21–99.
[4] N.N. Bogolyubov, Ju.A. Mitropolsky, A.M. Samoilenko, Methods of Accelerated Convergence in Nonlinear Mechanics, Springer, New York, 1976.
[5] H. Cendra, J.E. Marsden, T.S. Ratiu, Lagrangian reduction by stages, 1999, in press.
[6] Yu.N. Fedorov, V.V. Kozlov, Various aspects of *n*-dimensional rigid body dynamics, Amer. Math. Soc. Transl. 168 (1995) 141–171.
[7] M.J. Field, Local structure for equivariant dynamics, in: Singularity Theory and its Applications, Lecture Notes in Mathematics, vol. 1463, Springer, Berlin, 1991, pp. 142–166.
[8] A. Jorba, R. Ramírez-Ros, J. Villanueva, Effective reducibility of quasi-periodic linear equations close to constant coefficients, SIAM J. Math. Anal. 28 (1997) 178–188.
[9] R.A. Johnson, G.R. Sell, Smoothness of spectral subbundles and reducibility of quasi-periodic linear differential systems, J. Differential Equations 41 (1981) 262–288.
[10] V.V. Kozlov, Invariant measures of the Euler–Poincaré equations on Lie algebras, Funct. Anal. Appl. 22 (1988) 69–70.

[11] J. Hermans, A symmetric sphere rolling on a surface, Nonlinearity 8 (1995) 493–515.
[12] J.E. Marsden, Lectures on Mechanics, London Mathematical Society Lecture Note Series, vol. 174, Cambridge University Press, Cambridge, 1992.
[13] J.E. Marsden, J. Scheurle, Pattern evocation and geometric phases in mechanical systems with symmetry, Dynamics Stability Systems 10 (1995) 315–338.
[14] A.S. Mischenko, A.T. Fomenko, General Liouville method of integration of Hamiltonian systems, Funct. Anal. Appl. 12 (1978) 46–56.
[15] D.V. Zenkov, A.M. Bloch, J.E. Marsden, The energy-momentum method for stability of nonholonomic systems, Dynamics Stability Systems 13 (1998) 123–165.